FREE DIFFERENTIABLE ACTIONS OF S^1 AND S^3 ON HOMOTOPY 11-SPHERES

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1. **Introduction.** By a homotopy n-sphere we mean a closed differentiable n-manifold having the homotopy of S^n. In [6] Montgomery and Yang have shown that there are exactly 10 homotopy 7-spheres not diffeomorphic to one another such that on each of them there is a free differentiable S^1 action. Moreover, on any homotopy 7-sphere, if there exists a free differentiable S^1 action, then there are infinitely many topologically distinct from one another. It is interesting to study an analogue for the free differentiable actions of S^1 and S^3 on the homotopy 11-spheres. The first example of free differentiable S^1 and S^3 action on an exotic homotopy 11-sphere is due to the Hsiangs [4]. In this paper, we succeed in finding out all possible homotopy 11-spheres which admit free differentiable S^3 actions. More precisely, we shall prove the following results.

THEOREM A. *Let Σ_M^{11} denote the Milnor sphere which is the generator of θ_{11} [5]. Then the homotopy sphere $k\Sigma_M^{11}$, k odd, admits no free differentiable S^1 actions.*

THEOREM B. *A homotopy 11-sphere Σ^{11} admits a free differentiable S^3 action if and only if $\Sigma^{11} \approx 32k\Sigma_M^{11}$ for some $k \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 10, \pm 11, \pm 12, \pm 14, \pm 16 \pmod{31}$. These all admit infinitely many topologically distinct actions which can be distinguished by the first Pontrjagin class of the orbit spaces.*

Throughout the paper, Z denotes the ring of integers; Q denotes the field of rational numbers. We let QP^2 be the quaternionic projective plane, and let CP^5 be the complex projective 5-space, all having the usual differentiable structure.

The main methods used to prove these results are based on the computation of the Eells-Kuiper μ-invariant [2] and the Montgomery-Yang ν-invariant [6].

2. **Proof of Theorem A.** First, we recall the definition of ν-invariant. Let Σ be a homotopy $(4k-1)$-sphere which is the boundary of a compact, oriented, differentiable $4k$-manifold W. The invariant $\nu(\Sigma)$ of Montgomery-Yang [6] is defined as follows: We assume that

(1) there is an element $\beta \in H^2(W; Z)$ whose reduction mod 2 is the second Stiefel-Whitney class $w_2(W)$ of W. Note that the homomorphism

$$j^*: H^q(W, \Sigma; Q) \to H^q(W; Q)$$

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induced by the inclusion map is an isomorphism onto for $q=4l$, $0<l<k$ and for $q=2k$. Define $\nu(\Sigma)$ by

$$\nu(\Sigma) \equiv \left\langle e^{j^{*-1}\beta/2} \sum_{i=0}^{k-1} \hat{A}_i(p_1(W),\ldots,p_i(W))+N_k(p_1(W),\ldots,p_{k-1}(W)), [W]\right\rangle$$
$$+t_k\tau(W) \pmod 1,$$

where we use the notation in [2].

We now proceed with the proof of the Theorem A. In order to prove Theorem A it suffices to prove the following lemma because the Eells-Kuiper invariant $\mu(\Sigma_M^{11})\equiv -1/992 \pmod 1$ and $\theta_{11}=Z_{992}$ [5].

LEMMA 2.1. *Let S^1 act freely and differentiably on a homotopy 11-sphere Σ^{11}. Then*

(2) $\mu(\Sigma^{11})\equiv(18i+204i^2+576i^3+296j+1440ij+496t)/992 \pmod 1$, *for some integers j, i and t.*

Proof. Let α be the generator of $H^2(\Sigma^{11}/S^1;Z)$. Since Σ^{11} admits free S^1 action, we have the principal bundle $\xi\colon S^1\to\Sigma^{11}\to\Sigma^{11}/S^1$ which is homotopically equivalent to the classical Hopf bundle $S^1\to S^{11}\to CP^5$. Hence the total Pontrjagin class $p(\xi)$ of the bundle ξ is given by $p(\xi)=1+\alpha^2$. Let W be the total space of the associated disk bundle $D^2\to W\xrightarrow{\pi}\Sigma^{11}/S^1$ of ξ. The space W is homotopically equivalent to Σ^{11}/S^1 with $w_2(W)\neq 0$, and index $\tau(W)=1$. Let

$$p_1(\Sigma^{11}/S^1) = r_1\alpha^2, \qquad p_2(\Sigma^{11}/S^1) = r_2\alpha^4,$$
$$p_1(W) = r_3\beta^2, \qquad p_2(W) = r_4\beta^4,$$

where $\beta=\pi^*\alpha$, and $r_1, r_2, r_3, r_4\in Z$. We can see that $r_3=r_1+1$, and $r_4=r_1+r_2$ because

$$p^*(W) = \pi^*\{(1+r_1\alpha^2+r_2\alpha^4)(1+\alpha^2)\} = 1+r_3\beta^2+r_4\beta^4.$$

Since β reduction mod 2 is $w_2(W)$, the invariant $\nu(\Sigma^{11})$ is well defined. By substituting the above data into the formula for ν, we obtain

$$\nu(\Sigma^{11}) \equiv \frac{1}{2^{10}\cdot 3^2\cdot 5}-\frac{r_3}{2^{10}\cdot 3^2}+\frac{-4r_4+7r_3^2}{2^{10}\cdot 3^2\cdot 5}+\frac{1260r_3r_4-945r_3^3}{2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 31}-\frac{1}{2^7\cdot 31} \pmod 1.$$

But $\nu(\Sigma^{11})=2\mu(\Sigma^{11})$ [6]; hence

$$\begin{aligned}496\nu(\Sigma^{11}) &\equiv 992\mu(\Sigma^{11})\\ &\equiv \{-329-155r_3-124r_4+217r_3^2+60r_3r_4-45r_3^3\}/2^5\cdot 3^2\cdot 5 \pmod{496}\\ &\equiv \{-312+80r_1-64r_2+142r_1^2+60r_1r_2-45r_1^3\}/2^5\cdot 3^2\cdot 5 \pmod{496}.\end{aligned}$$

As μ is a complete invariant for homotopy 11-spheres, $992\mu(\Sigma^{11})$ is an integer modulo 496. Thus we must have $r_1=2s$ for some integer s, and

$$992\mu(\Sigma^{11}) \equiv \{-39+20s-8r_2+71s^2+15r_2s-45s^3\}/2^2\cdot 3^2\cdot 5 \pmod{496}.$$

Again we can see that both r_2 and s must be odd, say, $r_2=2h+1$ and $s=2n+1$ for some integers h and n. Then

$$992\mu(\Sigma^{11}) \equiv \{7+42n+7h-128n^2+30hn-180n^3\}/2\cdot 3^2\cdot 5 \pmod{496}.$$

The integer h must be odd, that is, $h=2a+1$ for some integer a. Hence we have

$$(3)\qquad 992\mu(\Sigma^{11}) \equiv \{7+36n+7a-64n^2+30an-90n^3\}/45 \pmod{496}.$$

On the other hand 2α (resp. 4α) reduction mod 2 is equal to $0=w_2(\Sigma^{11}/S^1)$, hence

$$A(\Sigma^{11}/S^1, \alpha) = \langle e^{\alpha}\hat{A}(\Sigma^{11}/S^1), [\Sigma^{11}/S^1]\rangle$$

(respectively, $A(\Sigma^{11}/S^1, 2\alpha)$) is an integer [3], where $[\Sigma^{11}/S^1]$ denotes the fundamental homology class of Σ^{11}/S^1. By a simple calculation

$$\begin{aligned} A(\Sigma^{11}/S^1, \alpha) &= \left\langle \frac{\alpha^5}{5!} - \frac{\alpha^3 p_1(\Sigma^{11}/S^1)}{6\cdot 24} + \frac{\alpha}{2^7\cdot 45}\{-4p_2(\Sigma^{11}/S^1)+7p_1^2(\Sigma^{11}/S^1)\}, [\Sigma^{11}/S^1]\right\rangle \\ &= \frac{1}{5!} - \frac{r_1}{6\cdot 24} + \frac{-4r_2+7r_1^2}{2^7\cdot 45} \\ &= (7n^2-3n-1-a)/360; \end{aligned}$$

here we use the fact that $r_1=4n+2$, and $r_2=4a+3$. Similarly

$$A(\Sigma^{11}/S^1, 2\alpha) = (7n^2-33n+29-a)/180.$$

Hence we have

$$(4)\qquad a \equiv 7n^2-3n-1 \pmod{360},$$

$$(5)\qquad a \equiv 7n^2-33n+29 \pmod{180}.$$

Solving (4) and (5) we obtain the solution $n\equiv 1 \bmod 6$, thus we can write

$$n = 6i+1, \qquad a = 7n^2-3n-1+360j$$

for some integers i and j. Substituting these values into (3), we get

$$992\mu(\Sigma^{11}) \equiv 18i+204i^2+576i^3+296j+1440ij \pmod{496}.$$

This completes the proof of Lemma 2.1.

Notice that $r_1=4n+2=24i+6$, hence as a by-product of the proof, we have the following corollary:

COROLLARY 2.2. *If S^1 acts freely and differentiably on a homotopy 11-sphere Σ^{11}, then*

$$p_1(\Sigma^{11}/S^1) = (24i+6)\alpha^2, \qquad p_2(\Sigma^{11}/S^1) = (1008i^2+264i+15+1440j)\alpha^4,$$

for some integers i and j.

3. **Proof of Theorem B.** First we will demonstrate a number of lemmas which lead to the proof of Theorem B.

LEMMA 3.1. *Let S^3 act freely and differentiably on a homotopy 11-sphere Σ^{11}. Then $\mu(\Sigma^{11}) \equiv n/124 \pmod 1$ for some integer n.*

Proof. The proof is almost the same as the proof of Lemma 2.1. We have the principal bundle $\eta: S^3 \to \Sigma^{11} \to \Sigma^{11}/S^3$ with associated disk bundle $D^4 \to W \xrightarrow{\pi} \Sigma^{11}/S^3$. The bundle η is homotopically equivalent to the Hopf bundle $S^3 \to S^{11} \to QP^2$, hence the total Pontrjagin class $p(\eta) = 1 + 2\bar{\alpha} + \bar{\alpha}^2$, where $\bar{\alpha}$ is the generator of $H^4(\Sigma^{11}/S^3; Z)$. Since W and Σ^{11}/S^3 are of the same homotopy type, it is not difficult to see that W is a spin manifold with index $\tau(W) = 0$. Let $p_1(\Sigma^{11}/S^3) = r\bar{\alpha}$, $r \in Z$, and $\beta = \pi^*\bar{\alpha} \in H^4(W; Z)$. By Hirzebruch's index theorem [3] we have

$$p_2(\Sigma^{11}/S^3) = (45+r^2)\bar{\alpha}^2/7.$$

The total Pontrjagin class $p(W)$ is given by

$$p(W) = \pi^*\{(1+r\bar{\alpha}+(45+r^2)\bar{\alpha}^2/7)(1+2\bar{\alpha}+\bar{\alpha}^2)\}.$$

This implies

$$p_1(W) = (r+2)\beta, \qquad p_2(W) = (r^2+14r+52)\beta^2/7.$$

We calculate the Eells-Kuiper μ-invariant

$$\begin{aligned}\mu(\Sigma^{11}) &\equiv \{4p_2(W)p_1(W) - 3p_1^3(W) - 24\tau\}[W]/2^{11}\cdot 3\cdot 31 \pmod 1 \\ &\equiv -(r+2)(r-2)(17r+62)/2^{11}\cdot 3\cdot 7\cdot 31 \pmod 1.\end{aligned}$$

The number $992\mu(\Sigma^{11})$ is an integer modulo 992, hence the integer r must be in the form $r = 4m+2$ for some integer m so that

$$992\mu(\Sigma^{11}) \equiv -m(m+1)(17m+24)/21 \pmod{992},$$

and

$$m(m+1)(17m+24) \equiv 0 \pmod{21}. \tag{6}$$

Note that $\hat{A}_2(\Sigma^{11}/S^3)$ is an integer because Σ^{11}/S^3 is a closed spin manifold [3]. As

$$\begin{aligned}\hat{A}_2(\Sigma^{11}/S^3) &= \{-4p_2(\Sigma^{11}/S^3) + 7p_1^2(\Sigma^{11}/S^3)\}/2^7\cdot 45 \\ &= (r^2-4)/2^7\cdot 7 = m(m+1)/56,\end{aligned}$$

hence

$$m(m+1) \equiv 0 \pmod{56}. \tag{7}$$

Thus there exists an integer c such that $m(m+1) = 56c$. Substituting this into (6) we obtain

$$8c(17m+24) \equiv 0 \pmod 3,$$

or $c(17m+24)/3$ is an integer. Therefore

$$\begin{aligned}\mu(\Sigma^{11}) &\equiv -m(m+1)(17m+24)/992\cdot 21 \pmod 1)\\ &\equiv -8\{c(17m+24)/3\}/992 \pmod 1)\\ &\equiv -\{c(17m+24)/3\}/124 \pmod 1).\end{aligned}$$

LEMMA 3.2. *If S^3 acts freely and differentiably on Σ^{11}, then $p_1(\Sigma^{11}/S^3)=(96\bar{\imath}+2)\bar{\alpha}$, for some $\bar{\imath}\in Z$, where $\bar{\alpha}$ is the generator of $H^4(\Sigma^{11}/S^3; Z)$.*

Proof. Since S^1 is a subgroup of S^3, the free action of S^3 on Σ^{11} induced a free action of S^1 on Σ^{11}. It follows from (2) and Lemma 3.1 that

$$18i+204i^2+576i^3+296j+1440ij \equiv 0 \pmod 8,$$

or

$$2i(9+102i) \equiv 0 \pmod 8.$$

Hence $i\equiv 0 \pmod 4$. This shows that $i=4\bar{\imath}$ for some integer $\bar{\imath}$.

Next consider the fibre bundle ζ: $S^2 \to \Sigma^{11}/S^1 \xrightarrow{\pi} \Sigma^{11}/S^3$ which is homotopically equivalent to the standard fibration $S^2 \to CP^5 \xrightarrow{\pi} QP^2$. By using the well-known fact that $p(\zeta)=1+4\bar{\alpha}$ [1],

$$p(\Sigma^{11}/S^1) = \pi^*\{[1+r\bar{\alpha}+(45+r^2)\bar{\alpha}^2/7][1+4\bar{\alpha}]\},$$

and so $r_1=r+4=24i+6=96\bar{\imath}+6$ by Corollary 2.2. Thus

$$r = 96\bar{\imath}+2. \tag{8}$$

Recall that $r=4m+2$, and m satisfies (6) and (7).

LEMMA 3.3. *The solutions m of* (6) *and* (7) *satisfying* (8) *are the following*:

$$m \equiv 0 \pmod{168}, \text{ hence } r \equiv 2 \pmod{672}, \tag{9}$$

or

$$m \equiv 48 \pmod{168}, \text{ hence } r \equiv 194 \pmod{672}. \tag{10}$$

The proof is an elementary number theory calculation.

Summing up, we have therefore proved

LEMMA 3.4. *Let S^3 act freely and differentiably on a homotopy* 11-*sphere Σ^{11}. Then one of the following holds*:

(11) $p_1(\Sigma^{11}/S^3)=(672k+2)\bar{\alpha}$ *for some integer k, and*

$$\mu(\Sigma^{11}) \equiv -(13k+1)(k+6)k/31 \pmod 1).$$

(12) $p_1(\Sigma^{11}/S^3)=(672k+194)\bar{\alpha}$ *for some integer k, and*

$$\mu(\Sigma^{11}) \equiv -7(-k+1)(7k+2)(k-7)/31 \pmod 1).$$

Now we can easily verify Theorem B. By direct computation, a free differentiable S^3 action on a homotopy 11-sphere Σ^{11} with $p_1(\Sigma^{11}/S^3)$ and $\mu(\Sigma^{11})$ satisfying (11) (respectively (12)) implies $\Sigma^{11} \approx 32k\Sigma_M^{11}$ for some $k \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 10, \pm 11, \pm 12, \pm 14, \pm 16 \pmod{31}$. But W. C. Hsiang and W. Y. Hsiang [4] have constructed examples satisfying (11) for all integers $k \in Z$. This clearly completes the proof of Theorem B.

It is of interest to find out what can be said about the existence of free differentiable S^3 actions on Σ^{11} satisfying (12).

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